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Prototype for time correlation effects: analytical results

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Abstract. We study a cellular automaton which is based on a Markovian process. The dynamics of the system is followed through the analysis of the associated Hamming distance as a function of time. The Hamming distance behaves in a peculiar plateau-like manner which exhibits the existence of memory in the time evolution of a damage. Consequently, this type of behaviour which has been found in complex systems does not necessarily arise from the complexity of the system. Analytical expressions are obtained for the probability of having finite-size plateaux as well as for the probability distribution of the size of the plateaux.

1. Introduction

Time correlation effects are present in a huge number of complex dynamical systems. One way to characterize the presence of correlation effects is to study the spreading of a damage introduced in the system. This may be achieved through the analysis of the time evolution of two different replicas of the system presenting a small difference, for instance, in the initial conditions or in the random sequence which generates the evolution of the system. In order to characterize the damage, the quantity whose time evolution is typically followed is the Hamming distance, see for instance [1]. This quantity behaves, as a function of time, in a more or less noise-like manner, like the quantities which characterize complex systems such as the discrete sandpile model [2].

A simple model, based on a Markovian process, was recently introduced by Tsallis *et al* [3] as a good prototype for correlation effects on damage. By means of computational simulations, they showed that the time evolution of the associated Hamming distance presents sequences from noise-like to plateau-like ones, for different values of the external parameters. In the present work, we derive an analytical expression for the probability of finding finite-size plateaux which confirms the results obtained previously through numerical simulations [3]. Moreover, we also describe analytically the distribution of probabilities ($P(\tau)$) of finding plateaux of size τ and the momenta of $P(\tau)$.

The analytical results for this simple model show that a non-trivial behaviour such as the presence of plateaux in the Hamming distance (behaviour indeed found in systems with self-organized criticality as the sandpile model in [2]) is not necessarily related to complexity and it can arise even in a simple Markovian-like process like the one herein focused. Given the current importance of complex systems, it is relevant to be able to determine whether a given behaviour results from the complexity of the system or not.

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2. The prototype

As described in [3], we assume a semi-infinite linear chain of sites ($i = 0, 1, 2, \ldots$) occupied by binary random variables $\{S_i\}$ ($S_i = 0, 1, \forall i$). The chain is defined through the following construction rules:

- $S_0 = 1$
- $S_{i+1} = S_i$ with probability p (thus $S_{i+1} \neq S_i$ with probability $1 - p$), for all $i = 0, 1, 2, \ldots$.

We consider two equivalent replicas of the system $\{S_i^A\}$ and $\{S_i^B\}$, constructed with the same value of p but generated with different random sequences.

In order to study the time dynamics of the damage evolution in the system, we define the following Hamming distance:

$$H(t) = \frac{1}{L} \sum_{i=i_0}^{i_0+L-1} |S_i^A - S_i^B| \tag{1}$$

where L is the length of the window we will analyse, $i_0 \equiv Jt$, J being a fixed positive integer number and t the discrete time ($t = 0, 1, 2, \ldots$). Then, the prototype can be interpreted as a cellular automaton of size L. In fact, once the chain configuration is defined by the construction rules, we look at windows of size L and the temporal evolution of the automaton is given by the fact that the window starts at a point i_0 which increases linearly with time.

The time evolution of the Hamming distance for different values of the external parameters is shown in figure 1. As already discussed in [3], the Hamming distance $H(t)$ as a function of time fluctuates exhibiting plateaux. Thus, it is specially interesting to analyse the behaviour of the distribution of plateaux present in the fluctuations of the Hamming distance. We say that there is a plateau at time t if $H(t) = H(t+1)$ and the plateau is of length τ if $H(t-1) \neq H(t) = H(t+1) = \cdots = H(t+\tau-1) = H(t+\tau) \neq H(t+\tau+1)$. For fixed values of the external variables (p, J, L), $H(t)$ yields a distribution of plateaux $P(\tau)$ ($\sum_{\tau=0}^{\infty} P(\tau) = 1$)) and the probability of having finite-size plateaux is $M(p, J, L) \equiv 1 - P(0)$.

3. Results

The probability distribution of the binary random variable S_i is

$$P(S_i) = p_i \delta(S_i - 1) + [1 - p_i]\delta(S_i) \tag{2}$$

where p_i is the probability of being $S_i = 1$. It is easy to find that p_i verifies the recurrence relation $p_i = (2p-1)p_{i-1} + 1 - p$, with $p_\circ = 1$, whose solution is $p_i = ((2p-1)^i + 1)/2$.

From equation (2), the mean value of S_i is $\langle S_i \rangle = p_i$, therefore, it follows directly from equation (1) that the mean value of the Hamming distance is

$$\langle H(t) \rangle = \frac{1}{2} - \frac{(2p-1)^{2Jt}}{8Lp(1-p)}(1 - (2p-1)^{2L}). \tag{3}$$

So that, in the limit $L \to \infty$, for $0 < p < 1$, we have:

$$\lim_{L \to \infty} \langle H(t) \rangle = \tfrac{1}{2} \tag{4}$$

hence, there are no long-range correlation effects for any value of J and $0 < p < 1$.

For a more detailed study, we want to know the probability of having finite size plateaux, $M(p, J, L)$, which is the probability that $H(t) = H(t+1)$ for an arbitrary time t. From

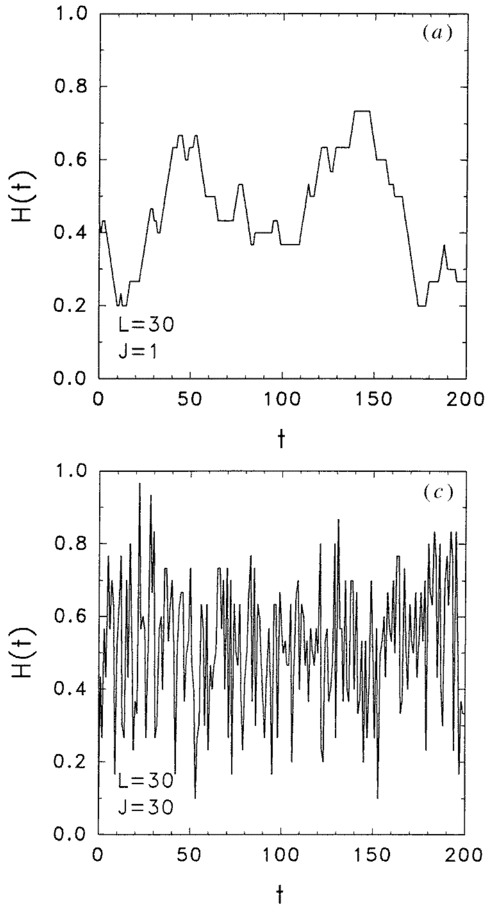

Figure 1. Temporal evolution of the Hamming distance for $p = 0.9$, $L = 30$ and $J = 1$ (*a*), 2 (*b*) and 30 (*c*).

the analysis of all possible configurations producing $H(t) = H(t+1)$ (see the appendix), we obtain an analytical expression for $M(p, J, L)$:

$$M(p, J, L) = M(R, a, b) = \tfrac{1}{2} R^{2(a-1)}(1 + W_1(R, a) + (2R-1)^{b-a+1}[1 + W_2(R, a)]) \tag{5}$$

where $a \equiv \min\{J, L\}$, $b \equiv \max\{J, L\}$ and $R \equiv R(p) = p^2 + (1-p)^2$,

$$W_1(R, a) = \tfrac{1}{2} \sum_{l=1}^{a-1} (\sigma_3(R, a, l) + 2\sigma_2(R, a, l) + \sigma_1(R, a, l))^2 \tag{6}$$

and

$$W_2(R, a) = \tfrac{1}{2} \sum_{l=1}^{a-1} (\sigma_3(R, a, l) - \sigma_1(R, a, l))^2 \tag{7}$$

with

$$\sigma_1(R,a,l) = \sum_{k=1}^{l-1} \binom{l-1}{k} \binom{a-l-1}{k-1} \left(\frac{1-R}{R}\right)^{2k} \tag{8}$$

$$\sigma_2(R,a,l) = \sum_{k=0}^{l-1} \binom{l-1}{k} \binom{a-l-1}{k} \left(\frac{1-R}{R}\right)^{2k+1} \tag{9}$$

$$\sigma_3(R,a,l) = \sum_{k=1}^{l} \binom{l-1}{k-1} \binom{a-l-1}{k} \left(\frac{1-R}{R}\right)^{2k}. \tag{10}$$

The comparison of these results with the experimental ones as obtained in [3] shows an excellent agreement within the standard deviation of the values from simulations. For a wide range of values of the external parameters, the maximal percentual difference between theoretical and experimental values of M was of the order of 1%.

From equations (5)–(10), in the particular case $p = \frac{1}{2}$ (when $\frac{1-R}{R} = 1$), by means of the property:

$$\sum_{i=0}^{q} \binom{m}{i} \binom{n}{q-i} = \binom{m+n}{q}$$

we obtain:

$$M(R = \tfrac{1}{2}, a, b) = \frac{1}{4^a} \binom{2a}{a} \tag{11}$$

which corresponds to the result obtained, through another treatment, in [3], in the discussion of the completely random case ($p = \frac{1}{2}$).

If $p = 0, 1$, then $M = 1$, $\forall J, L$.

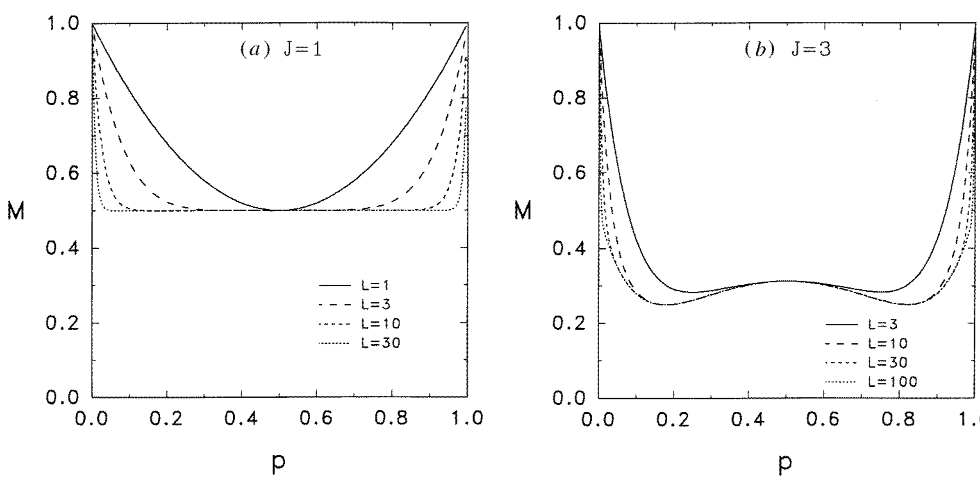

Figure 2. $M(p, J, L)$ versus p as obtained from equation (5). (a) $J = 1$ and different values of L; (b) $J = 3$ and different values of $L \geqslant J$. Values of L are indicated on the figure. Scales are the same in both graphs for comparison.

For $J = 1$ (figure 2(a)), M has a local minimum at $p_{\rm min} = \frac{1}{2}$, being $M_{\rm min} = M(p_{\rm min}) = \frac{1}{2}$, $\forall L \geqslant 1$. For $0 < p < 1$ and $p \neq \frac{1}{2}$, M decreases for increasing L, i.e. the concavity of the curve increases, such that as $L \to \infty$, $M \to \frac{1}{2}$, for $0 < p < 1$.

For $1 < J \leqslant L$ (illustrated in figure 2(*b*)), M has a local maximum at $p = \frac{1}{2}$, and two absolute minima at $p_{\min}$ and $1 - p_{\min}$, such that, for a given value of J, $M_{\max}$ is independent on L while $p_{\min}$ and $M_{\min} = M(p_{\min}) = M(1 - p_{\min})$ decrease as L increases down to a constant value. This value decreases for increasing values of J. In the limit $J \to \infty$ (hence $L \to \infty$), $M_{\max} \to \frac{1}{\sqrt{\pi J}} \to 0$, $p_{\min} \to 0$, $M_{\min} \to 0$. Thus, when p approaches either zero or unity, correlation persists for increasing values of J, while correlation effects disappear ($M \to 0$) for $J \to \infty$ (hence, $L \to \infty$).

Other cases are reduced to the previous ones recalling that M is a symmetrical function of J and L, i.e. $M(p, J, L) = M(p, L, J)$.

In sum, we observe that M varies gradually with the model parameters. Therefore, it does not play the role of an order parameter, in contrast with the remark in [3].

Finally, we want to determine the probability distribution $P(\tau)$, for $0 < p < 1$. From the preceding discussion we note that the probability of having a plateau at time t, that is $M = \mathrm{pr}(H(t) = H(t+1))$, is independent of t for any value of the external parameters. Recalling also that there is a plateau of length τ if $H(t-1) \neq H(t) = H(t+1) = \cdots = H(t+\tau-1) = H(t+\tau) \neq H(t+\tau+1)$, then it follows that:

$$P(\tau) = \tau(1-M)^2 M^\tau \qquad \text{for } \tau \geqslant 1 \tag{12}$$

which verifies $\sum_{\tau=1}^{\infty} P(\tau) = M$.

The mean size of the plateaux is $\langle\tau\rangle = \sum_{\tau=0}^{\infty} \tau P(\tau)$, then, we have:

$$\langle\tau\rangle = \frac{M(1+M)}{1-M}. \tag{13}$$

Straightforwardly, the nth momenta of $P(\tau)$ ($\langle\tau^n\rangle$) may be obtained from the following recurrence relation derived from equation (12):

$$\langle\tau^n\rangle = M\frac{\partial\langle\tau^{n-1}\rangle}{\partial M} + \frac{2M}{1-M}\langle\tau^{n-1}\rangle \qquad n > 1. \tag{14}$$

Solving equation (14), we obtain:

$$\langle\tau^n\rangle = \frac{M}{(1-M)^n}\mathcal{P}_n(M) \tag{15}$$

where $\mathcal{P}_n(M)$ is a polynomial of order n in the variable M whose coefficients may be obtained recursively from the triangle in table 1.

In particular, after calculating the square deviation σ_τ^2, we get the ratio

$$\frac{\langle\tau\rangle}{\sigma_\tau} = \frac{1+M}{\sqrt{2+(1-M)(1+M)^2/M}}. \tag{16}$$

The distribution of probabilities of the plateau size, $P(\tau)$, presents a maximum at $\tau_{\max} \sim \frac{-1}{\ln M}$, then $\tau_{\max}$ increases with increasing M and so does the mean size of the plateaux $\langle\tau\rangle$. Above $\tau_{\max}$, $P(\tau)$ decreases following a power law with exponent τ. Its standard deviation is such that $0 \leqslant \frac{\langle\tau\rangle}{\sigma_\tau} \leqslant \sqrt{2}$, $\frac{\langle\tau\rangle}{\sigma_\tau}$ increasing for increasing M and being unitary for $M \sim 0.75$.

4. Discussion

The Hamming distance associated to the present prototype behaves as a function of time in a more or less noise-like manner (figure 1). From figure 2, we observe that the probability M of finding finite-size plateaux varies gradually with the model parameters, i.e. the behaviour of the system varies gradually from noise-like to plateau-like. Although the prototype

Table 1. Coefficients $c_{n,i}$ of the polynomial $\mathcal{P}_n(M) = \sum_{i=0}^{n} c_{n,i} M^i$. The coefficients of $\mathcal{P}_n$ are obtained from those of $\mathcal{P}_{n-1}$ by adding the latter multiplied by integer factors as indicated by the arrows.

	i									
n	0		1		2		3		4	
1	(1)		(1)							
	↓×1	↘×2	↓×2	↘×1						
2	(1)		(4)		(1)					
	↓×1	↘×3	↓×2	↘×2	↓×3	↘×1				
3	(1)		(11)		(11)		(1)			
	↓×1	↘×4	↓×2	↘×3	↓×3	↘×2	↓×4	↘×1		
4	(1)		(26)		(66)		(26)		(1)	
	↓×1	↘×5	↓×2	↘×4	↓×3	↘×3	↓×4	↘×2	↓×4	↘×1

is based on a typical Markov process, the introduction of the parameters J and L puts into evidence a non-trivial behaviour also observed in self-organized criticality. In fact, for appropriate values of the model parameters, it is possible to mimic the plateau-like behaviour of the Hamming distance associated to the discrete sandpile model [2] (compare figure 1(*b*) with figure 1 of [2]). Thus, that kind of behaviour is not necessarily related to complexity and may simply arise from some Markovian process involved. Our analytical results may help to understand the origin of such time correlation.

Connections between spread of damage and relevant thermal equilibrium quantities of discrete statistical models have been established recently [4–6]. In particular, for the Ising model, relations between the Hamming distance and some of their thermal quantities are found [4]. The relevance of this correspondence relies on the fact that they provide a new approach in order to calculate thermostatistic quantities. Therefore, the present results may lead to a better understanding of correlation effects in statistical systems and their consequences in relation to critical phenomena.

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Appendix. Determination of $M(p, J, L)$

In order to calculate the probability that $H(t) = H(t+1)$ for an arbitrary time t, let us rewrite equation (1) as

$$H(t) = \frac{1}{L} \sum_{i=Jt}^{Jt+L-1} h_i \tag{A1}$$

where $h_i = |S_i^A - S_i^B|$. Comparing $H(t)$ with $H(t+1)$, we only need to consider their non-common terms. For a systematic analysis, let us first study the special case $J = 1$ and later the wider case $J > 1$.

A.1. Case $J = 1$

In this case, the non-common terms are h_t and h_{t+L}, $\forall L$. Therefore:

$$M(p, 1, L) = \mathrm{pr}(h_t = h_{t+L}). \tag{A2}$$

We consider arrays $a_i = \begin{bmatrix} S_i^A \\ S_i^B \end{bmatrix}$ which represent the values of the random variable S_i on the two replicas A and B. The possible arrays are: $a^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $a^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $a^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $a^4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, from equation (A2), we have:

$$M(p, 1, L) = \sum_{\alpha,\beta=1}^{2} \mathrm{pr}(a_t = a^\alpha \wedge a_{t+L} = a^\beta) + \sum_{\alpha,\beta=3}^{4} \mathrm{pr}(a_t = a^\alpha \wedge a_{t+L} = a^\beta). \tag{A3}$$

By considering the prototype construction rules, it follows:

$$\mathrm{pr}(S_{i+k} = S_i) = (1 + (2p - 1)^k)/2 \qquad \forall\ i, k \geqslant 0 \tag{A4}$$

hence,

$$\mathrm{pr}(S_{i+k} \neq S_i) = (1 - (2p - 1)^k)/2.$$

From equation (A4) and taking into account that the values of S_{t+L} and S_t are not independent, we have:

$$p^{++} = \left(\frac{1 + (2p - 1)^L}{2}\right)^2 \tag{A5}$$

$$p^{--} = \left(\frac{1 - (2p - 1)^L}{2}\right)^2 \tag{A6}$$

$$p^{+-} = p^{-+} = \frac{1 - (2p - 1)^{2L}}{4} \tag{A7}$$

where p^{++}, p^{--} and p^{+-} (p^{-+}) are, respectively, the probabilities that either both, none or one of the elements of the arrays a_t and a_{t+L} be equal.

Therefore, by substituting equations (A5) and (A6) in equation (A3), we get:

$$M(p, 1, L) = (1 + (2p - 1)^{2L})/2. \tag{A8}$$

A.2. Case $J > 1$

For the case $J > 1$, in order to compare $H(t)$ with $H(t + 1)$, we consider two cases: (a) $L \geqslant J$ and (b) $1 \leqslant L \leqslant J$.

A.2.1. Subcase (a): $L \geqslant J$. Now, we must consider arrays of $2 \times J$ elements, e.g. $\begin{bmatrix} \overbrace{01 \cdots 1}^{J} \\ 11 \cdots 0 \end{bmatrix}$. In this case $H(t)$ and $H(t + 1)$ share $L - J$ terms. Thus, we must compare J terms: $(h_{Jt}, \ldots, h_{Jt+J-1})$ with $(h_{Jt+L}, \ldots, h_{Jt+L+J-1})$. In order to perform this comparison, we consider pairs of $2 \times J$-arrays corresponding to $[a_{Jt}, \ldots, a_{Jt+J-1}]$ versus $[a_{Jt+L}, \ldots, a_{Jt+L+J-1}]$.

For a given array, let us call l ($0 \leqslant l \leqslant J$) the number of columns formed by equal elements (i.e. either $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$). Since we are interested in the configurations giving

$H(t)$=$H(t+1)$, we have to consider only pairs of arrays of $2\times J$ elements with the same value of l which will have the same Hamming distance $J-l$. So that we calculate the contributions M_l corresponding to each value of l to the probability M. We have:

$$M_l = \mathrm{pr}(h_{Jt}+\cdots+h_{Jt+J-1} = h_{Jt+L}+\cdots+h_{Jt+L+J-1} = J-l)$$
$$= \sum_{\text{all arrays }(J,l)} \mathrm{pr}([a_{Jt},\ldots,a_{Jt+J-1}]_{J,l})$$
$$\times \mathrm{pr}([a_{Jt+L},\ldots,a_{Jt+L+J-1}]_{J,l}/[a_{Jt},\ldots,a_{Jt+J-1}]_{J,l}) \qquad \text{(A9)}$$

subindices J and l in the arrays indicate the total number of columns and the number of columns with equal elements, respectively. If incompatible with the form of the array, then pr(array$_{J,l}$)=0, e.g. $\mathrm{pr}\left(\begin{bmatrix}1\cdots0\\0\cdots0\end{bmatrix}_{J,J}\right)=0$, since in this case ($l=J$) all columns should have identical elements.

Besides, we have

$$\mathrm{pr}([a_{Jt},\ldots,a_{Jt+J-1}]_{J,l})\mathrm{pr}([a_{Jt+L},\ldots,a_{Jt+L+J-1}]_{J,l}/[a_{Jt},\ldots,a_{Jt+J-1}]_{J,l})$$
$$= \mathrm{pr}(a_{Jt})\mathrm{pr}([a_{Jt},\ldots,a_{Jt+J-1}]_{J,l})\mathrm{pr}([a_{Jt+L},\ldots,a_{Jt+L+J-1}]_{J,l})$$
$$\times \mathrm{pr}(a_{Jt+L}/a_{Jt+J-1}) \qquad \text{(A10)}$$

where the probabilities will be defined as follows:

• $\mathrm{pr}(a_{Jt})$ in equation (A10) is the probability that the first column of the first array (i.e. 2×1 array at position Jt) be equal to a given a^α, $1\leqslant\alpha\leqslant4$.

• The probability pr(array$_{J,l}$) in equation (A10), is the probability of having two given sequences of length J corresponding to rows A and B of the array. This probability is not that of finding the sequences at a given position of the samples but that of just having a given sequence without caring for the position. Thus, if we interchange all 1's in the array by 0's and vice versa, as well as if we interchange rows, or reverse the order of the columns, the probability is the same.

• In order to calculate the conditional probability $\mathrm{pr}(a_{Jt+L}/a_{Jt+J-1})$ in equation (A10), we must now consider the last column of the first array and the first column of the second array. Here, the 'distance' k between the arrays, that is the difference between the first site of the second array ($Jt+L$) and the last site of the first array ($Jt+J-1$), is $k=L-J+1$, so that we just have to replace L in equations (A5)–(A7) by $L-J+1$. Thus, let us define the following probabilities: P_J^+ is the probability that the last column of the first array and the first column of the second have two or no equal elements and P_J^- is the probability that those columns have only one equal element:

$$P_J^+ = p_J^{++}+p_J^{--} = (1+(2p-1)^{2(L-J+1)})/2$$
$$P_J^- = p_J^{+-}+p_J^{-+} = (1-(2p-1)^{2(L-J+1)})/2$$

where p_J^{++}, p_J^{--}, p_J^{+-} and p_J^{-+} were calculated in the same way as the probabilities in equations (A5)–(A7), corresponding to $J=1$, but substituting L by $L-J+1$.

Considering that the arrays with $l=\lambda$ and $l=J-\lambda$ (with $0\leqslant\lambda\leqslant J$) are obtained one from the other by interchanging 0's and 1's in one of the rows and also considering that $\sum_{\alpha=1}^4 \mathrm{pr}(a_{Jt}=a^\alpha)=1$, then, from equations (A9) and (A10), the contribution $M_{l,J-l}=M_l+M_{J-l}$ is:

$$M_{l,J-l} = 2P_J^-\,\mathrm{pr}\left(\begin{bmatrix}0\cdots\\0\cdots\end{bmatrix}_{J,l}\right)\mathrm{pr}\left(\begin{bmatrix}0\cdots\\1\cdots\end{bmatrix}_{J,l}\right)$$
$$+P_J^+\left(\left(\mathrm{pr}\left(\begin{bmatrix}0\cdots\\0\cdots\end{bmatrix}_{J,l}\right)\right)^2+\left(\mathrm{pr}\left(\begin{bmatrix}0\cdots\\1\cdots\end{bmatrix}_{J,l}\right)\right)^2\right) \qquad \text{(A11)}$$

where, there is an additional $\frac{1}{2}$ factor if $l = J - l$.

Since

$$\mathrm{pr}\left(\begin{bmatrix} 0\cdots \\ 1\cdots \end{bmatrix}_{J,l}\right) = \mathrm{pr}\left(\begin{bmatrix} 0\cdots \\ 0\cdots \end{bmatrix}_{J,J-l}\right)$$

then, we just need to know $\mathrm{pr}\left(\begin{bmatrix} 0\cdots \\ 0\cdots \end{bmatrix}_{J,l}\right)$.

⋆ For $l = J, 0$, from equation (A11), we obtain

$$M_{l=J,0} = P_J^+ \times \left(\mathrm{pr}\left(\begin{bmatrix} 0\cdots \\ 0\cdots \end{bmatrix}_{J,J}\right)\right)^2.$$

It is easy to see that

$$\mathrm{pr}\left(\begin{bmatrix} 0\cdots \\ 0\cdots \end{bmatrix}_{J,J}\right) = (p^2 + (1-p)^2)^{2(J-1)}$$

then, we have:

$$M_{l=J,0} = (1 + (2p-1)^{2(L-J+1)})/2(p^2 + (1-p)^2)^{2(J-1)}.$$

⋆ Now, let us calculate $\mathrm{pr}\left(\begin{bmatrix} 0\cdots \\ 0\cdots \end{bmatrix}_{J,l}\right)$ when, $l \neq J, 0$. Let us define

$$V_{J,l} = \mathrm{pr}\left(\begin{bmatrix} 0\cdots \\ 0\cdots \end{bmatrix}_{J,l}\right) \tag{A12}$$

$$W_{J,l} = \mathrm{pr}\left(\begin{bmatrix} 0\cdots \\ 1\cdots \end{bmatrix}_{J,l}\right) \tag{A13}$$

and

$$R = R(p) = p^2 + (1-p)^2. \tag{A14}$$

It is easy to see that:

$$V_{J,l} = RV_{J-1,l-1} + (1-R)W_{J-1,l-1} \tag{A15}$$

$$W_{J,l} = (1-R)V_{J-1,l} + RW_{J-1,l}. \tag{A16}$$

Then, we get the following recurrence relation:

$$V_{J,l} = RV_{J-1,l-1} + (1-R)V_{J-1,J-l} \tag{A17}$$

with solution:

$$V_{J,l} = R^{J-1}(\sigma_1(R, J, l) + \sigma_2(R, J, l)) \tag{A18}$$

where

$$\sigma_1(R, J, l) = \sum_{k=1}^{l-1} \binom{l-1}{k} \binom{J-l-1}{k-1} \left(\frac{1-R}{R}\right)^{2k} \tag{A19}$$

$$\sigma_2(R, J, l) = \sum_{k=0}^{l-1} \binom{l-1}{k} \binom{J-l-1}{k} \left(\frac{1-R}{R}\right)^{2k+1} \tag{A20}$$

thus,

$$W_{J,l} = V_{J,J-l} = R^{J-1}(\sigma_3(R, J, l) + \sigma_2(R, J, l)) \tag{A21}$$

where

$$\sigma_3(R, J, l) = \sigma_1(R, J, J-l) = \sum_{k=1}^{l} \binom{l-1}{k-1} \binom{J-l-1}{k} \left(\frac{1-R}{R}\right)^{2k}. \tag{A22}$$

Note also that $\sigma_2(R, J, J-l) = \sigma_2(R, J, l)$.

By substitution in equation (A11) we obtain:

$$M_{l,J-l} = \tfrac{1}{2}(V_{J,l} + W_{J,l})^2 + \tfrac{1}{2}(2p-1)^{2(L-J+1)}(V_{J,l} + W_{J,l})^2. \tag{A23}$$

Then:

$$M = M_{J,0} + \sum_{l=1}^{[\frac{J}{2}]-1} M_{l,J-l} + \tfrac{1}{2} M_{[\frac{J}{2}],[\frac{J}{2}]} = M_{J,0} + \tfrac{1}{2}\sum_{l=1}^{J-1} M_{l,J-l}. \tag{A24}$$

A.2.2. Subcase (b): $1 \leqslant L \leqslant J$. In this case we have two arrays of $2 \times L$ elements and $k = J - L + 1$. Since the calculation of M will only depend on the size of the arrays and on the 'distance' k, then case (b) is reduced to case (a) just by inverting the roles of J and L, being $M(p, J, L) = M(p, L, J)$.

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